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# Duals of curves in Hyperbolic space

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**Abstract.** We define two kinds of duals of a curve in Hyperbolic space and investigate the singularities and the relationship from the view point of Legendrian dualities.

**Keywords:** Hyperbolic space, Legendrian dualities, curves

**MSC 2010 classification:** Primary 53A35, Secondary 58C27

## 1 Introduction

In this paper we investigate curves in Hyperbolic 3-space from the view point of dual relations. For a curve in Hyperbolic space with non-zero hyperbolic curvature, we define a *de Sitter dual surface* of the curve in de Sitter space which is the natural analogue of the dual surface of a curve in Euclidean 3-sphere. We give a classification of the singularities of de Sitter dual surface (§4) and investigate the geometric meanings (§5). On the other hand, there exists another dual surface in the lightcone which is called a *horospherical surface* of the curve [2]. In §3 we give a relationship between those dual surfaces of the curve from the view point of Legendrian dualities which were introduced in [3].

## 2 Basic notions and results

We adopt the model of the hyperbolic 3-space in the Lorentz-Minkowski space-time. Let  $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3) \mid x_i \in \mathbb{R} \ (i = 0, 1, 2, 3)\}$  be a 4-dimensional vector space. For any  $\mathbf{x} = (x_0, x_1, x_2, x_3)$ ,  $\mathbf{y} = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$ , the *pseudo scalar product* of  $\mathbf{x}$  and  $\mathbf{y}$  is defined by  $\langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + \sum_{i=1}^3 x_iy_i$ . We call

$(\mathbb{R}^4, \langle, \rangle)$  *Lorentz-Minkowski space-time*. We denote  $\mathbb{R}_1^4$  instead of  $(\mathbb{R}^4, \langle, \rangle)$ . We say that a non-zero vector  $\mathbf{x} \in \mathbb{R}_1^4$  is *spacelike*, *lightlike* or *timelike* if  $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ ,  $\langle \mathbf{x}, \mathbf{x} \rangle = 0$  or  $\langle \mathbf{x}, \mathbf{x} \rangle < 0$  respectively. For a vector  $\mathbf{v} \in \mathbb{R}_1^4$  and a real number  $c$ , we define the *hyperplane with pseudo normal  $\mathbf{v}$*  by  $HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^4 \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}$ . We call  $HP(\mathbf{v}, c)$  a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if  $\mathbf{v}$  is timelike, spacelike or lightlike respectively.

We now define *Hyperbolic 3-space* by

$$H_+^3(-1) = \{\mathbf{x} \in \mathbb{R}_1^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_0 \geq 1\},$$

*de Sitter 3-space* by

$$S_1^3 = \{\mathbf{x} \in \mathbb{R}_1^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$$

and a *closed lightcone* with the vertex  $\mathbf{a}$  by

$$LC_a = \{\mathbf{x} \in \mathbb{R}_1^4 \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0\}.$$

We denote that  $LC_+^* = \{\mathbf{x} = (x_0, x_1, x_2, x_3) \in LC_0 \mid x_0 > 0\}$  and we call it the *future lightcone* at the origin. For any  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}_1^4$ , we define a vector  $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$  by

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 = \begin{vmatrix} -\mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix},$$

where  $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  is the canonical basis of  $\mathbb{R}_1^4$ . We have three kinds of surfaces in  $H_+^3(-1)$  which are given by intersections of  $H_+^3(-1)$  and hyperplanes in  $\mathbb{R}_1^4$ . A surface  $H_+^3(-1) \cap HP(\mathbf{v}, c)$  is called a *sphere*, an *equidistant surface* or a *horosphere* if  $H(\mathbf{v}, c)$  is spacelike, timelike or lightlike respectively. We write  $SP^2(\mathbf{v}, c)$  as a sphere and  $ES^2(\mathbf{v}, c)$  as an equidistant surface. Especially,  $ES^2(\mathbf{v}, 0)$  is called a *hyperbolic plane*.

We now construct the explicit differential geometry on curves in  $H_+^3(-1)$ . Let  $\gamma : I \rightarrow H_+^3(-1)$  be a regular curve. Since  $H_+^3(-1)$  is a Riemannian manifold, we can reparametrise  $\gamma$  by the arc-length. Hence, we may assume that  $\gamma(s)$  is a unit speed curve. So we have the tangent vector  $\mathbf{t}(s) = \gamma'(s)$  with  $\|\mathbf{t}(s)\| = 1$ , where  $\|\mathbf{v}\| = \sqrt{|\langle \mathbf{v}, \mathbf{v} \rangle|}$ . In the case when  $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle \neq -1$ , then we have a unit vector  $\mathbf{n}(s) = \frac{\mathbf{t}'(s) - \gamma(s)}{\|\mathbf{t}'(s) - \gamma(s)\|}$ . Moreover, define  $\mathbf{e}(s) = \gamma(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}(s)$ , then we have a pseudo orthonormal frame  $\{\gamma(s), \mathbf{t}(s), \mathbf{n}(s), \mathbf{e}(s)\}$  of  $\mathbb{R}_1^4$  along  $\gamma$ . By standard arguments, under the assumption that  $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle \neq -1$ , we have the

following *Frenet-Serret type formula*:

$$\begin{cases} \gamma'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = \kappa_h(s)\mathbf{n}(s) + \gamma(s), \\ \mathbf{n}'(s) = -\kappa_h(s)\mathbf{t}(s) + \tau_h(s)\mathbf{e}(s), \\ \mathbf{e}'(s) = -\tau_h(s)\mathbf{n}(s), \end{cases}$$

where  $\kappa_h(s) = \|\mathbf{t}'(s) - \gamma(s)\|$  and  $\tau_h(s) = -\frac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{(\kappa_h(s))^2}$ .

Since  $\langle \mathbf{t}'(s) - \gamma(s), \mathbf{t}'(s) - \gamma(s) \rangle = \langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle + 1$ , the condition  $\langle \mathbf{t}'(s), \mathbf{t}'(s) \rangle \neq -1$  is equivalent to the condition  $\kappa_h(s) \neq 0$ . We can study all properties of hyperbolic space curves by using this natural equation.

Let  $\gamma : I \rightarrow H_+^3(-1)$  be a unit speed hyperbolic space curve with  $\kappa_h \neq 0$ . We define a map as follows:

$$DD_\gamma : I \times J \rightarrow S_1^3; DD_\gamma(s, \theta) = \cos \theta \mathbf{n}(s) + \sin \theta \mathbf{e}(s)$$

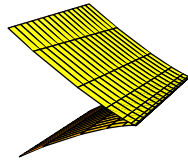
where  $0 \leq \theta < 2\pi$ , which is called a *de Sitter dual surface* of  $\gamma$ ,

In this paper we give a classification of the singularities of this surface.

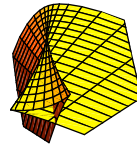
**Theorem 1.** *Let  $\gamma : I \rightarrow H_+^3(-1)$  be a unit speed hyperbolic space curve with  $\kappa_h \neq 0$ . Then we have the following:*

- (1) *The de Sitter dual surface  $DD_\gamma$  of  $\gamma$  is singular at a point  $(s_0, \theta_0)$  if and only if  $\theta_0 = \pi/2$  or  $\theta_0 = 3\pi/2$ .*
- (2) *The de Sitter dual surface  $DD_\gamma$  of  $\gamma$  is locally diffeomorphic to the cuspidal edge  $C \times \mathbb{R}$  at  $(s_0, \theta_0)$  if  $\theta_0 = \pi/2$  or  $\theta_0 = 3\pi/2$  and  $\tau_h(s_0) \neq 0$ .*
- (3) *The de Sitter dual surface  $DD_\gamma$  of  $\gamma$  is locally diffeomorphic to the swallow tail  $SW$  at  $(s_0, \theta_0)$  if  $\theta_0 = \pi/2$  or  $\theta_0 = 3\pi/2$ ,  $\tau_h(s_0) = 0$  and  $\tau_h'(s_0) \neq 0$ .*

Here,  $C \times \mathbb{R} = \{(x_1, x_2, x_3) | x_1^2 = x_2^3\}$  is the *cuspidal edge* (cf. Fig.1) and  $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$  is the *swallow tail* (cf. Fig.2).



cuspidaledge  
Fig.1.



swallowtail  
Fig. 2.

The geometric meanings of the singularities of  $DD_\gamma$  and  $\tau_h$  are given in §5.

On the other hand, the *horospherical surface* of  $\gamma$  is defined as follows [2]:

$$HS_\gamma : I \times J \longrightarrow LC^*; HS_\gamma(s, \theta) = \gamma(s) + \cos \theta \mathbf{n}(s) + \sin \theta \mathbf{e}(s).$$

In order to characterize the singularities of horospherical surface, a hyperbolic invariant  $\sigma_h(s)$  is defined to be

$$\sigma_h(s) = ((\kappa'_h)^2 - (\kappa_h)^2(\tau_h)^2((\kappa_h)^2 - 1))(s).$$

The singularities of the horospherical surfaces are classified into the following theorem.

**Theorem 2.** *[[2]] Let  $\gamma : I \longrightarrow H^3_+(-1)$  be a unit speed hyperbolic space curve with  $\kappa_h \neq 0$ . Then we have the following:*

- (1) *The horospherical surface  $HS_\gamma$  of  $\gamma$  is singular at a point  $(s_0, \theta_0)$  if and only if  $\cos \theta_0 = 1/\kappa_h(s_0)$ .*
- (2) *The horospherical surface  $HS_\gamma$  of  $\gamma$  is locally diffeomorphic to the cuspidal edge  $C \times \mathbb{R}$  at  $(s_0, \theta_0)$  if  $\cos \theta_0 = 1/\kappa_h(s_0)$  and  $\sigma_h(s_0) \neq 0$ .*
- (3) *The horospherical surface  $HS_\gamma$  of  $\gamma$  is locally diffeomorphic to the swallow tail SW at  $(s_0, \theta_0)$  if  $\cos \theta_0 = 1/\kappa_h(s_0)$ ,  $\sigma_h(s_0) = 0$  and  $\sigma'_h(s_0) \neq 0$ .*

### 3 Legendrian dualities

In [3] the second author introduced the Legendrian dualities between pseudo-spheres in Lorentz-Minkowski space. We require some properties of contact manifolds and Legendrian submanifolds for the duality results in this paper. Let  $N$  be a  $(2n+1)$ -dimensional smooth manifold and  $K$  be a field of tangent hyperplanes on  $N$ . Such a field is locally defined by a 1-form  $\alpha$ . The tangent hyperplane field  $K$  is said to be *non-degenerate* if  $\alpha \wedge (d\alpha)^n \neq 0$  at any point on  $N$ . The pair  $(N, K)$  is a *contact manifold* if  $K$  is a non-degenerate hyperplane field. In this case  $K$  is called a *contact structure* and  $\alpha$  a *contact form*. A submanifold  $i : L \subset N$  of a contact manifold  $(N, K)$  is said to be *Legendrian* if  $\dim L = n$  and  $di_x(T_x L) \subset K_{i(x)}$  at any  $x \in L$ . A smooth fibre bundle  $\pi : E \rightarrow M$  is called a *Legendrian fibration* if its total space  $E$  is furnished with a contact structure and the fibers of  $\pi$  are Legendrian submanifolds. Let  $\pi : E \rightarrow M$  be a Legendrian fibration. For a Legendrian submanifold  $i : L \subset E$ ,  $\pi \circ i : L \rightarrow M$  is called a *Legendrian map*. The image of the Legendrian map  $\pi \circ i$  is called a *wavefront set* of  $i$  and is denoted by  $W(i)$ .

The duality concepts we use in this paper are those introduced in [3], where four Legendrian double fibrations are considered on the subsets  $\Delta_i$ ,  $i = 1, \dots, 4$  of the product of two of the pseudo spheres  $H^n(-1)$ ,  $S^n_1$  and  $LC^*$ . In this paper we need the following three Legendrian double fibrations:

- (1) (a)  $H^3(-1) \times S_1^3 \supset \Delta_1 = \{(v, w) \mid \langle v, w \rangle = 0\}$ ,  
 (b)  $\pi_{11} : \Delta_1 \rightarrow H^3(-1)$ ,  $\pi_{12} : \Delta_1 \rightarrow S_1^3$ ,  
 (c)  $\theta_{11} = \langle dv, w \rangle|_{\Delta_1}$ ,  $\theta_{12} = \langle v, dw \rangle|_{\Delta_1}$ .
- (2) (a)  $H^3(-1) \times LC^* \supset \Delta_2 = \{(v, w) \mid \langle v, w \rangle = -1\}$ ,  
 (b)  $\pi_{21} : \Delta_2 \rightarrow H^3(-1)$ ,  $\pi_{22} : \Delta_2 \rightarrow LC^*$ ,  
 (c)  $\theta_{21} = \langle dv, w \rangle|_{\Delta_2}$ ,  $\theta_{22} = \langle v, dw \rangle|_{\Delta_2}$ .
- (3) (a)  $LC^* \times S_1^3 \supset \Delta_3 = \{(v, w) \mid \langle v, w \rangle = 1\}$ ,  
 (b)  $\pi_{31} : \Delta_3 \rightarrow LC^*$ ,  $\pi_{32} : \Delta_3 \rightarrow S_1^3$ ,  
 (c)  $\theta_{31} = \langle dv, w \rangle|_{\Delta_3}$ ,  $\theta_{32} = \langle v, dw \rangle|_{\Delta_3}$ .

Above,  $\pi_{i1}(v, w) = v$  and  $\pi_{i2}(v, w) = w$  for  $i = 1, 2, 3$ ,  $\langle dv, w \rangle = -w_0 dv_0 + \sum_{i=1}^3 w_i dv_i$  and  $\langle v, dw \rangle = -v_0 dw_0 + \sum_{i=1}^3 v_i dw_i$ . The 1-forms  $\theta_{i1}$  and  $\theta_{i2}$ ,  $i = 1, 2, 3$ , define the same tangent hyperplane field over  $\Delta_i$  which is denoted by  $K_i$ . We have the following duality theorem on the above spaces.

**Theorem 3.** *[[3]] The pairs  $(\Delta_i, K_i)$ ,  $i = 1, 2, 3$ , are contact manifolds and  $\pi_{i1}$  and  $\pi_{i2}$  are Legendrian fibrations.*

Given a Legendrian submanifold  $i : L \rightarrow \Delta_i$ ,  $i = 1, 2, 3$ , We say that  $\pi_{i1}(i(L))$  is the  $\Delta_i$ -dual of  $\pi_{i2}(i(L))$  and vice-versa. Then we have the following dual relations on de Sitter duals and horospherical surfaces.

**Theorem 4.** *Let  $\gamma : I \rightarrow H_+^3(-1)$  be a unit speed hyperbolic space curve with  $\kappa_h \neq 0$ . Then we have the following:*

- (1)  $\gamma$  is the  $\Delta_1$ -dual of  $DD_\gamma$ .
- (2)  $\gamma$  is the  $\Delta_2$ -dual of  $HS_\gamma$ .
- (3)  $HS_\gamma$  is the  $\Delta_3$ -dual of  $DD_\gamma$ .

*Proof.* (1) Consider a mapping  $\mathcal{L}_1 : I \times J \rightarrow \Delta_1$  defined by  $\mathcal{L}_1(s, \theta) = (\gamma(s), DD_\gamma(s, \theta))$ . Then we have  $\langle \gamma(s), DD_\gamma(s, \theta) \rangle = 0$ , so that the mapping is well-defined. Since we have

$$\frac{\partial \mathcal{L}_1}{\partial s}(s, \theta) = (t(s), \cos \theta n'(s) + \sin \theta e'(s)), \quad \frac{\partial \mathcal{L}_1}{\partial \theta}(s, \theta) = (0, -\sin \theta n(s) + \cos \theta e(s)),$$

$\mathcal{L}_1$  is an immersion. Moreover, we have  $\mathcal{L}_1^* \theta_{11} = \langle t(s), \cos \theta n(s) + \sin \theta e(s) \rangle = 0$ . Therefore,  $\mathcal{L}_1(I \times J)$  is a Legendrian submanifold in  $\Delta_1$ .

(2) We define a mapping  $\mathcal{L}_2 : I \times J \rightarrow \Delta_2$  by  $\mathcal{L}_2(s, \theta) = (\gamma(s), HS_\gamma(s, \theta))$ . We also define a mapping  $\Psi_{12} : \Delta_1 \rightarrow \Delta_2$  by  $\Psi_{12}(v, w) = (v, v + w)$ . We can easily show that this mapping is well-defined. Moreover, we have  $(\Psi_{12})^* \theta_{21} = \langle dv, v + w \rangle = \langle dv, w \rangle = \theta_{11}$ . We have the inverse mapping  $\Psi_{21} : \Delta_2 \rightarrow \Delta_1$  defined by  $\Psi_{21}(v, w) = (v, w - v)$ . Thus,  $\Psi_{12}$  is a contact diffeomorphism from

$\Delta_1$  to  $\Delta_2$ . By definition, we have  $\Psi_{12} \circ \mathcal{L}_1 = \mathcal{L}_2$ , so that  $\mathcal{L}_2(I \times J)$  is a Legendrian submanifold in  $\Delta_2$ .

(3) We define a mapping  $\Psi_{13} : \Delta_1 \longrightarrow \Delta_3$  by  $\Psi_{13}(\mathbf{v}, \mathbf{w}) = (\mathbf{v} + \mathbf{w}, \mathbf{w})$ . By the similar calculation to the case (2), we can show that  $\Psi_{13}$  is a contact diffeomorphism from  $\Delta_1$  to  $\Delta_3$ . By definition, we have  $\Psi_{13} \circ \mathcal{L}_1 = (HS_\gamma, DD_\gamma)$ , so that  $(HS_\gamma, DD_\gamma)(I \times J)$  is a Legendrian submanifold in  $\Delta_3$ . This completes the proof.  $\square$

## 4 De Sitter height functions

In this section we introduce a family of functions on a curve which is useful for the study of invariants of hyperbolic space curves. For a hyperbolic space curve  $\gamma : I \longrightarrow H_+^3(-1)$ , we define a function  $D : I \times S_1^3 \longrightarrow \mathbb{R}$  by  $D(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle$ . We call  $D$  a *de Sitter height function* on  $\gamma$ . We denote that  $d_{v_0}(s) = D(s, \mathbf{v}_0)$  for any  $\mathbf{v}_0 \in S_1^3$ . Then we have the following proposition.

**Proposition 1.** *Let  $\gamma : I \longrightarrow H_+^3(-1)$  be a unit speed hyperbolic space curve with  $\kappa_h \neq 0$ . Then we have the following:*

- (1)  $d_{v_0}(s_0) = 0$  if and only if there exist  $\lambda, \mu, \nu \in \mathbb{R}$  such that  $\mathbf{v}_0 = \lambda \mathbf{t}(s_0) + \mu \mathbf{n}(s_0) + \nu \mathbf{e}(s_0)$ .
- (2)  $d_{v_0}(s_0) = d'_{v_0}(s_0) = 0$  if and only if  $\mathbf{v}_0 = \cos \theta \mathbf{n}(s_0) + \sin \theta \mathbf{e}(s_0)$ , where  $\theta \in [0, 2\pi)$ .
- (3)  $d_{v_0}(s_0) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = 0$  if and only if  $\mathbf{v}_0 = \pm \mathbf{e}(s_0)$ .
- (4)  $d_{v_0}(s_0) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = 0$  if and only if  $\tau_h(s_0) = 0$  and  $\mathbf{v}_0 = \pm \mathbf{e}(s_0)$ .
- (5)  $d_{v_0}(s_0) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = d^{(4)}_{v_0}(s_0) = 0$  if and only if  $\tau_h(s_0) = \tau'_h(s_0) = 0$  and  $\mathbf{v}_0 = \pm \mathbf{e}(s_0)$ .

*Proof.* Since  $d_{v_0}(s) = \langle \gamma(s), \mathbf{v}_0 \rangle$ , we have the following calculations:

- (a)  $d'_{v_0}(s) = \langle \mathbf{t}(s), \mathbf{v}_0 \rangle$ ,
- (b)  $d''_{v_0}(s) = \langle \kappa_h(s) \mathbf{n}(s) + \gamma(s), \mathbf{v}_0 \rangle$ ,
- (c)  $d'''_{v_0}(s) = \langle (1 - (\kappa_h)^2(s)) \mathbf{t}(s) + \kappa'_h(s) \mathbf{n}(s) + \kappa_h(s) \tau_h(s) \mathbf{e}(s), \mathbf{v}_0 \rangle$ ,
- (d)  $d^{(4)}_{v_0}(s) = \langle (1 - (\kappa_h)^2(s)) \gamma(s) - 3\kappa_h(s) \kappa'_h(s) \mathbf{t}(s) + (\kappa_h(s) - \kappa_h^3(s) - \kappa_h(s) (\tau_h)^2(s) + \kappa''_h(s) \mathbf{n}(s) + (2\kappa'_h(s) \tau_h(s) + \kappa_h(s) \tau'_h(s)) \mathbf{e}(s), \mathbf{v}_0 \rangle$ .

By the definition of the de Sitter height function, the assertion (1) follows. By the formula (a),  $d_{v_0}(s) = d'_{v_0}(s_0) = 0$  if and only if  $\mu^2 + \nu^2 = 1$ . It follows that  $\mu = \cos \theta, \nu = \sin \theta$ , where  $0 \leq \theta < 2\pi$ . Therefore the assertion (2) holds. By the formula (b),  $d_{v_0}(s_0) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = 0$  if and only if

$\kappa_h(s)\cos\theta = 0$ . Since  $\kappa_h(s_0) \neq 0$ , we have  $\theta = \pi/2, 3\pi/2$ . We have the assertion (3). By the formula (c),  $d_{v_0}(s) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = 0$  if and only if  $\tau_h(s) = 0$  and  $\mathbf{v}_0 = \pm \mathbf{e}(s_0)$ . This means that the assertion (4) holds. By the similar arguments to the above, we can show the assertion (5). This completes the proof.  $\square$

In order to prove Theorem 1, we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [1]. Let  $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow \mathbb{R}$  be a function germ. We call  $F$  an  $r$ -parameter unfolding of  $f$ , where  $f(s) = F_{x_0}(s, x_0)$ . We say that  $f$  has an  $A_k$ -singularity at  $s_0$  if  $f^{(p)}(s_0) = 0$  for all  $1 \leq p \leq k$ , and  $f^{(k+1)}(s_0) \neq 0$ . We also say that  $f$  has an  $A_{\geq k}$ -singularity at  $s_0$  if  $f^{(p)}(s_0) = 0$  for all  $1 \leq p \leq k$ . Let  $F$  be an unfolding of  $f$  and  $f(s)$  has an  $A_k$ -singularity ( $k \geq 1$ ) at  $s_0$ . We denote the  $(k-1)$ -jet of the partial derivative  $\partial F / \partial x_i$  at  $s_0$  by  $j^{(k-1)}(\partial F / \partial x_i(s, x_0))(s_0) = \sum_{j=0}^{k-1} \alpha_{ji}(s-s_0)^j$  for  $i = 1, \dots, r$ . Then  $F$  is called a *versal unfolding* if the  $k \times r$  matrix of coefficients  $(\alpha_{ji})$  has rank  $k$  ( $k \leq r$ ).

We now introduce an important set concerning the unfoldings relative to the above notions. The *discriminant set* of  $F$  is the set

$$\mathcal{D}_F = \left\{ x \in \mathbb{R}^r \mid \text{there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x) \right\}.$$

Then we have the following well-known result (cf., [1]).

**Theorem 5.** *Let  $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow \mathbb{R}$  be an  $r$ -parameter unfolding of  $f(s)$  which has an  $A_k$  singularity at  $s_0$ . Suppose that  $F$  is a versal unfolding.*

- (1) *If  $k = 2$ , then  $\mathcal{D}_F$  is locally diffeomorphic to  $C \times \mathbb{R}^{r-2}$ .*
- (2) *If  $k = 3$ , then  $\mathcal{D}_F$  is locally diffeomorphic to  $SW \times \mathbb{R}^{r-3}$ .*

For the proof of Theorem 1, we have the following proposition.

**Proposition 2.** *Let  $\gamma : I \rightarrow H_+^3(-1)$  be a unit speed hyperbolic space curve with  $\kappa_h \neq 0$  and  $D : I \times S_1^3 \rightarrow \mathbb{R}$  be the de Sitter height function on  $\gamma(s)$ . If  $d_{v_0}$  has an  $A_k$ -singularity ( $k = 2, 3$ ) at  $s_0$ , then  $D$  is a versal unfolding of  $d_{v_0}$ .*

*Proof.* Let us consider the pseudo orthonormal basis  $\mathbf{e}_0 = \gamma(s_0)$ ,  $\mathbf{e}_1 = \mathbf{t}(s_0)$ ,  $\mathbf{e}_2 = \mathbf{n}(s_0)$  and  $\mathbf{e}_3 = \mathbf{e}(s_0)$  instead of the canonical basis of  $\mathbb{R}_1^4$ . Then

$$D(s, \mathbf{v}) = -v_0 x_0(s) + v_1 x_1(s) + v_2 x_2(s) + v_3 x_3(s),$$

where  $v_i$  and  $x_i(s)$  denote respectively the coordinates of  $\mathbf{v}$  and  $\gamma(s)$  with respect

to this basis. Since  $v_3 = \sqrt{v_0^2 - v_1^2 - v_2^2 + 1}$ , we have

$$\begin{aligned}\frac{\partial D}{\partial v_0}(s, \mathbf{v}) &= -x_0(s) + \frac{v_0}{v_3}x_3(s), \quad \frac{\partial^2 D}{\partial s \partial v_0}(s, \mathbf{v}) = -x'_0(s) + \frac{v_0}{v_3}x'_3(s), \\ \frac{\partial^3 D}{\partial s^2 \partial v_0}(s, \mathbf{v}) &= -x''_0(s) + \frac{v_0}{v_3}x''_3(s), \\ \frac{\partial D}{\partial v_i}(s, \mathbf{v}) &= x_i(s) - \frac{v_i}{v_3}x_3(s), \quad \frac{\partial^2 D}{\partial s \partial v_i}(s, \mathbf{v}) = x'_i(s) - \frac{v_i}{v_3}x'_3(s), \\ \frac{\partial^3 D}{\partial s^2 \partial v_i}(s, \mathbf{v}) &= x''_i(s) - \frac{v_i}{v_3}x''_3(s), \quad (i = 1, 2),\end{aligned}$$

so that we consider the following matrix:

$$A = \begin{pmatrix} -x_0(s_0) + \frac{v_0}{v_3}x_3(s_0) & x_1(s_0) - \frac{v_1}{v_3}x_3(s_0) & x_2(s_0) - \frac{v_2}{v_3}x_3(s_0) \\ -x'_0(s_0) + \frac{v_0}{v_3}x'_3(s_0) & x'_1(s_0) - \frac{v_1}{v_3}x'_3(s_0) & x'_2(s_0) - \frac{v_2}{v_3}x'_3(s_0) \\ -x''_0(s_0) + \frac{v_0}{v_3}x''_3(s_0) & x''_1(s_0) - \frac{v_1}{v_3}x''_3(s_0) & x''_2(s_0) - \frac{v_2}{v_3}x''_3(s_0) \end{pmatrix}.$$

We denote that

$$\mathbf{a}_i = \begin{pmatrix} x_i(s_0) \\ x'_i(s_0) \\ x''_i(s_0) \end{pmatrix}, \quad (i = 0, 1, 2, 3).$$

Then we have

$$\begin{aligned}\det A &= \frac{v_0}{v_3} \det(\mathbf{a}_3 \ \mathbf{a}_1 \ \mathbf{a}_2) + \frac{v_1}{v_3} \det(\mathbf{a}_0 \ \mathbf{a}_3 \ \mathbf{a}_2) + \frac{v_2}{v_3} \det(\mathbf{a}_0 \ \mathbf{a}_1 \ \mathbf{a}_3) - \frac{v_3}{v_3} \det(\mathbf{a}_0 \ \mathbf{a}_1 \ \mathbf{a}_2) \\ &= \frac{v_0}{v_3} \det(\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3) - \frac{v_1}{v_3} \det(\mathbf{a}_0 \ \mathbf{a}_2 \ \mathbf{a}_3) + \frac{v_2}{v_3} \det(\mathbf{a}_0 \ \mathbf{a}_1 \ \mathbf{a}_3) - \frac{v_3}{v_3} \det(\mathbf{a}_0 \ \mathbf{a}_1 \ \mathbf{a}_2).\end{aligned}$$

Since we have

$$\gamma \wedge \gamma' \wedge \gamma'' = (-\det(\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3), -\det(\mathbf{a}_0 \ \mathbf{a}_2 \ \mathbf{a}_3), +\det(\mathbf{a}_0 \ \mathbf{a}_1 \ \mathbf{a}_3), -\det(\mathbf{a}_0 \ \mathbf{a}_1 \ \mathbf{a}_2))$$

$$\text{at } s = s_0, \det A = \langle (\frac{v_0}{v_3}, \frac{v_1}{v_3}, \frac{v_2}{v_3}, \frac{v_3}{v_3}), (\gamma' \wedge \gamma'' \wedge \gamma''') \rangle = \langle \frac{1}{v_3} \mathbf{e}(s_0), \kappa_h(s_0) \mathbf{e}(s_0) \rangle =$$

$$\frac{\kappa_h(s_0)}{v_3} \neq 0. \text{ Thus, we have rank } A = 3.$$

If we consider the matrix

$$B = \begin{pmatrix} -x_0(s_0) + \frac{v_0}{v_3}x_3(s_0) & x_1(s_0) - \frac{v_1}{v_3}x_3(s_0) & x_2(s_0) - \frac{v_2}{v_3}x_3(s_0) \\ -x'_0(s_0) + \frac{v_0}{v_3}x'_3(s_0) & x'_1(s_0) - \frac{v_1}{v_3}x'_3(s_0) & x'_2(s_0) - \frac{v_2}{v_3}x'_3(s_0) \end{pmatrix},$$

this consists of the first and the second columns of the matrix  $A$ , so that the rank of  $B$  is two. If  $d_{v_0}$  has an  $A_k$ -singularity ( $k=2,3$ ) at  $s_0$ , then  $D$  is a versal unfolding of  $d_{v_0}$ . This completes the proof.  $\square$



*Proof.* (*Proof of Theorem 1.*) By Proposition 1, (2), the discriminant set  $\mathcal{D}_D$  of the de Sitter height function  $D$  of  $\gamma$  is the image of the de Sitter dual surface of  $\gamma$ . The singularities of the discriminant set are corresponding to the points of Proposition 1, (3), so that the assertion (1) holds. It also follows from Proposition 1, (4) and (5), that  $d_{v_0}$  has the  $A_2$ -type singularity (respectively, the  $A_3$ -type singularity) at  $s = s_0$  if and only if  $\theta_0 = 2/\pi, 3\pi/2$  and  $\tau_h(s_0) \neq 0$  (respectively,  $\theta_0 = \pi/2, 3\pi/2$  and  $\tau_h(s_0) = 0, \tau'_h(s_0) \neq 0$ ). By Theorem 5 and Proposition 2, we have the assertions (2) and (3). This completes the proof.  $\square$

## 5 Invariants of hyperbolic space curves

In this section we investigate the geometric properties of the singularities of  $DD_\gamma$  by using the invariant  $\tau_h$  of  $\gamma$ . At first, we consider the case when  $\tau_h \equiv 0$ .

**Proposition 3.** *Let  $\gamma : I \rightarrow H_+^3(-1)$  be a unit speed hyperbolic space curve with  $\kappa_h \neq 0$ . For the de Sitter dual surface  $DD_\gamma(s, \theta) = \cos\theta \mathbf{n}(s) + \sin\theta \mathbf{e}(s)$  of  $\gamma$  and  $\theta_0 = \pi/2, 3\pi/2$ , the following conditions are equivalent:*

- (a)  $DD_\gamma(s, \theta_0)$  is a constant vector,
- (b)  $\tau_h(s) \equiv 0$ ,
- (c)  $\text{Im}(\gamma) \subset ES^2(\mathbf{v}, 0)$  for a spacelike vector  $\mathbf{v}$ .

*Proof.* Suppose that  $\theta_0 = \pi/2, 3\pi/2$ . Then we have  $DD_\gamma(s, \theta_0) = \pm \mathbf{e}(s)$  and  $\frac{\partial DD_\gamma(s, \theta_0)}{\partial s} = \mp \tau_h(s) \mathbf{e}(s)$ , so that  $\frac{dDD_\gamma(s, \theta_0)}{ds}(s) \equiv 0$  if and only if  $\tau_h(s) \equiv 0$ . This means that the condition (a) is equivalent to the condition (b). Suppose that  $\tau_h(s) \equiv 0$ . Then  $DD_\gamma(s, \theta_0) = \pm \mathbf{e}(s) = \pm \mathbf{v}$  are constant. Since  $\langle \gamma(s), \pm \mathbf{e}(s) \rangle = 0$ ,  $\text{Im}(\gamma) \subset H_+^3(-1) \cap HP(\mathbf{v}, 0)$ . Here,  $\mathbf{e}(s) = \mathbf{v}$  is spacelike, so that  $HP(\mathbf{v}, 0)$  is timelike.

On the other hand, suppose that  $\text{Im}(\gamma) \subset H_+^3(-1) \cap HP(\mathbf{v}, 0)$  and  $\mathbf{v}$  is spacelike. Then we have  $h_v(s) = \langle \gamma(s), \mathbf{v} \rangle = 0$ . By Proposition 1, (4),  $\tau_h(s) \equiv 0$ . This completes the proof  $\square$

The above proposition asserts that the degeneracy of singularities of  $DD_\gamma$  might relates how the curve contact with a hyperbolic plane. Let  $F : H_+^3(-1) \rightarrow \mathbb{R}$  be a submersion and  $\gamma : I \rightarrow H_+^3(-1)$  be a spacelike curve. We say that  $\gamma$  has *k-point contact* with  $F^{-1}(0)$  at  $t = t_0$  if the function  $g(t) = F \circ \gamma(t)$  satisfies  $g(t_0) = g'(t_0) = \dots = g^{(k-1)}(t_0) = 0$ ,  $g^{(k)}(t_0) \neq 0$ . We also say that  $\gamma$  has *at least k-point contact* with  $F^{-1}(0)$  at  $t = t_0$  if the function  $g(t) = F \circ \gamma(t)$  satisfies  $g(t_0) = g'(t_0) = \dots = g^{(k-1)}(t_0) = 0$ . We now consider a function  $\mathcal{D} : H_+^3(-1) \times S_1^3 \rightarrow \mathbb{R}$  defined by  $\mathcal{D}(\mathbf{x}, \mathbf{v}) = \langle \mathbf{x}, \mathbf{v} \rangle$ . Then we have

$D(s, \mathbf{v}) = \mathcal{D} \circ (\gamma \times 1_{S^3_1})$ . Thus, we have the following proposition as a corollary of Proposition 1.

**Proposition 4.** *For  $\mathbf{v}_0 = DD_\gamma(s_0, \theta_0)$ , we have the following:*

- (1)  $\gamma$  has at least 2-point contact with  $ES(\mathbf{v}_0, 0)$  at  $s_0$  if and only if  $\theta_0 = \pi/2$  or  $3\pi/2$ .
- (2)  $\gamma$  has 3-point contact with  $ES(\mathbf{v}_0, 0)$  at  $s_0$  if and only if  $\theta_0 = \pi/2$  or  $3\pi/2$  and  $\tau_h(s_0) \neq 0$ .
- (3)  $\gamma$  has 4-point contact with  $ES(\mathbf{v}_0, 0)$  at  $s_0$  if and only if  $\theta_0 = \pi/2$  or  $3\pi/2$ ,  $\tau_h(s_0) = 0$  and  $\tau'_h(s_0) \neq 0$ .

By Theorem 1, we have the following geometric characterization of the singularities of  $DD_\gamma$  as follows:

**Theorem 6.** *Let  $\gamma : I \rightarrow H^3_+(-1)$  be a unit speed hyperbolic space curve with  $\kappa_h \neq 0$ . For  $\mathbf{v}_0 = DD_\gamma(s_0, \theta_0)$ , we have the following:*

- (1) *The de Sitter dual surface  $DD_\gamma$  is singular at a point  $(s_0, \theta_0)$  if and only if  $\gamma$  has at least 2-point contact with  $ES(\mathbf{v}_0, 0)$  at  $s_0$ .*
- (2) *The de Sitter dual surface  $DD_\gamma$  of  $\gamma$  is locally diffeomorphic to the cuspidal edge  $C \times \mathbb{R}$  at  $(s_0, \theta_0)$  if  $\gamma$  has 3-point contact with  $ES(\mathbf{v}_0, 0)$  at  $s_0$ .*
- (3) *The de Sitter dual surface  $DD_\gamma$  of  $\gamma$  is locally diffeomorphic to the swallow tail  $SW$  at  $(s_0, \theta_0)$  if  $\gamma$  has 4-point contact with  $ES(\mathbf{v}_0, 0)$  at  $s_0$ .*

## References

- [1] J. W. BRUCE AND P. J. GIBLIN: *Curves and singularities* (second edition), Cambridge University press, 1992
- [2] S. IZUMIYA, D.-H. PEI AND T. SANO: *Horospherical surfaces of curves in hyperbolic space*, Publ. Math. Debrecen, **64** (2004), 1–13.
- [3] S. IZUMIYA: *Legendrian dualities and spacelike hypersurfaces in the lightcone*, Moscow Mathematical Journal, **9** (2009), 325–357.